

8. THE ALEXANDER NUMBER OF A KNOT

§8.1. A Primary School Level Account

I'm now going to introduce the Alexander number of a knot. This can be done at several levels – the first level can be, and has been, successfully understood by students in upper primary school. It's all done with buttons.

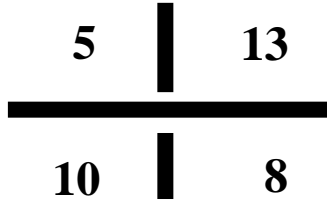


You first explain to the students about knots having both ends tied together and explain how you can draw a picture of them, with the convention of the broken line to indicate which part goes underneath. (No mention need be made of multiple crossings.)

It might be fun to get the students to colour these pictures of knots using only two colours, though this has nothing to do with their understanding of how to find Alexander's number for the knot.

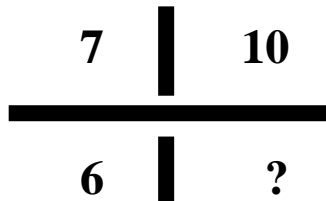
You then explain what's meant by a 'crossing', perhaps get them to count crossings, and you explain that at each crossing there are four spaces. (There's no need to mention the special case where there are only three regions.)

You then explain that we have to put buttons in each of the spaces so that the total number of buttons on each side of the unbroken line are the same. For example:



That is, $5 + 13 = 18 = 10 + 8$.

You then explain that if we have buttons in three out of the four spaces we can work out what has to go in the remaining one. For example if we have:



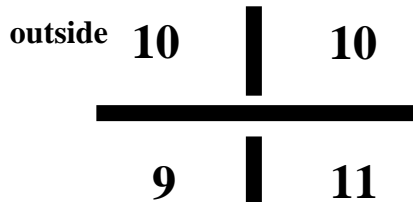
we can deduce that the fourth space must have 11 buttons since $7 + 10 = 6 + 11$.

We can call such crossings ‘balanced’. You can practise the students’ arithmetic by getting them to balance some crossings by filling in the missing fourth number. At this stage crossings should be taken in isolation and you give them three out of the four numbers.

Also, the missing number shouldn't be negative. This, after all, is a primary school level account.

Once the students are proficient at balancing crossings you let them loose on balancing a whole knot. You begin by putting 10 buttons in the outside, near a crossing. (There's no need to use real buttons – primary school children will be able to cope with the abstraction of representing 10 buttons by the number 10.)

On the other side of the broken line you put another 10 and in the spaces on the other side you put 9 and 11 (it doesn't matter which is 9 and which is 11). We now have one balanced crossing (with a total of 20 buttons on each side of the unbroken line).



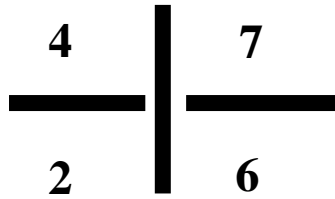
Now you look for another crossing where three out of the four spaces have numbers. (Make sure you emphasise that you can only have one number in each space. So the whole of the outside is 10 and this 10 can be used to balance other crossings that involve the outside.) You then balance this crossing.

In this way you go around the knot, balancing crossings. Some crossings may only have two of its spaces numbered. These can't be balanced yet. But later,

when we've balanced other crossings we can come back to them because we may now have a third space numbered.

Now listen very carefully. When you get to the very last crossing you'll find that all four of its spaces have been numbered, but it won't balance! Don't panic. It doesn't mean that you've made a mistake.

For example you might find that when you've balanced all but one of the crossings, the last crossing looks like this.

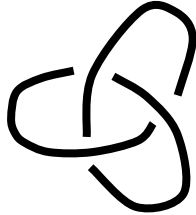


On one side we've got $4 + 2 = 6$ while on the other side we've got $7 + 6 = 13$.

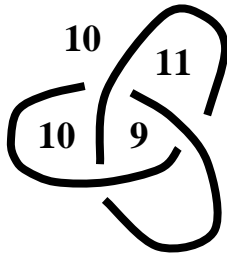
To balance the very last crossing you'll need to use some 'magic' buttons that you keep off to one side somewhere. You use these to balance that last crossing. In this case we need to add an extra 7 magic buttons to one side to make the 6 up to 13. We don't actually change any of the numbers in the knot picture because this will unbalance crossings we've already balanced. It's just that we use an extra number to add to one side of the last balancing equation to get the other. In this case that

special number would be 7. And this would be the Alexander number for the knot.

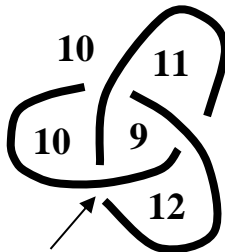
Example 1:



We start by putting down our first balanced crossing: 10, 10, 9 and 11.

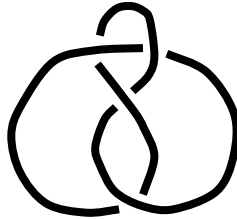


Now we balance the crossing on the right. (Remember that the whole of the outside is 10 so on one side we have $11 + 10 = 21$.)

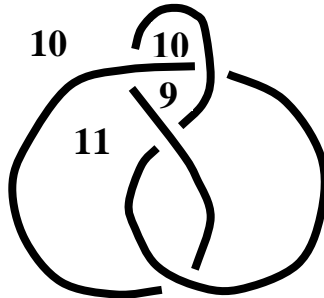


But at the crossing at the bottom we've got $10 + 9 = 19$ on one side and $10 + 12 = 22$ on the other. We need to borrow three magic buttons to make it balance. This means that the Alexander number for this knot is 3.

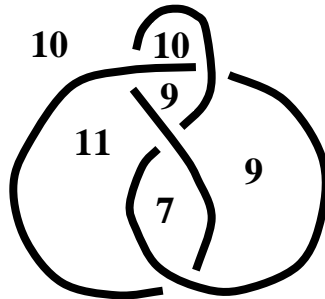
Example 2:



We begin with our 10, 10, 9 and 11 balanced crossing.

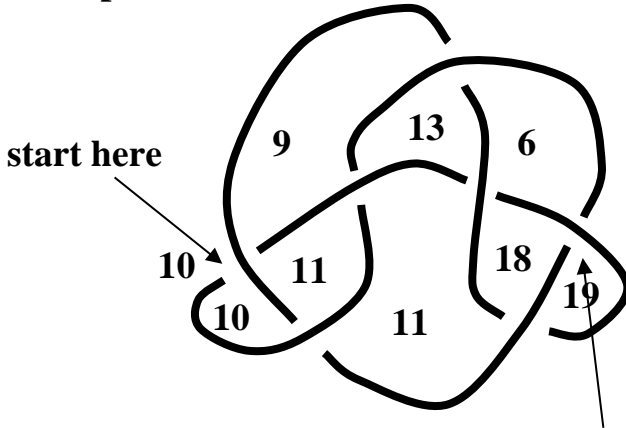


Then, balancing all but the bottom one we get:



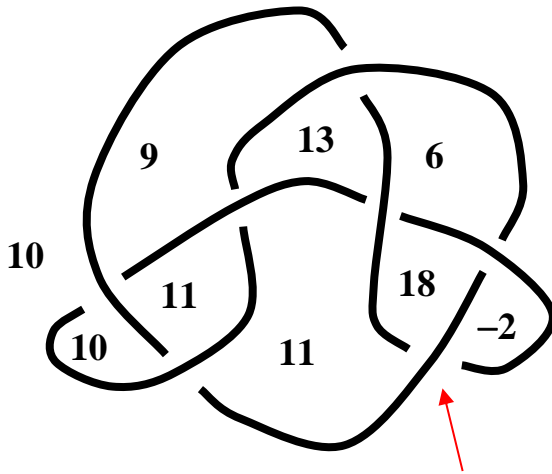
As usual, the last crossing is unbalanced: $11 + 10 = 21$ on one side and $7 + 9 = 16$ on the other. The discrepancy is 5, and this is the Alexander number of the figure eight knot.

Example 3:



The discrepancy at the crossing indicated is 21, so this is the Alexander number of this knot.

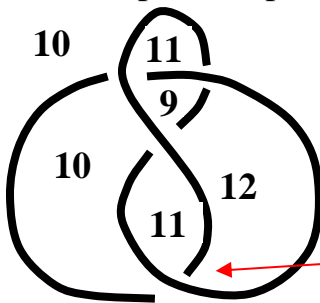
Notice that we had to be a little clever to avoid using negative numbers. If we'd balanced the crossing indicated, instead of leaving it till last, we'd have had to fill in -2 in the space at the right instead of 19. But we'd still have the discrepancy at the bottom right crossing of 21.



Here the discrepancy is 21.

So, by choosing a different order for the crossings you can get different numbers in the spaces. But the discrepancy will still come out the same.

Example 4: This knot looks like the figure 8 knot, whose Alexander number we've seen is 5. But on closer inspection it can be deformed into one of the trefoil knots, with Alexander number 3. Let's see if we still get 3 with this more complicated picture of the trefoil knot.



The discrepancy here is 3.

This illustrates (but doesn't prove) the fact that the Alexander number is an invariant. But it is the case that you can tangle up a knot, making it more complicated, and its Alexander number won't change.

With more complicated knots we run the risk of having to use negative numbers. We got very close to this in example 3. Of course there's nothing wrong with using negative numbers. It's just that for presenting this method to primary aged students one should avoid them. However there's quite a large number of simple knots we can use at this level.

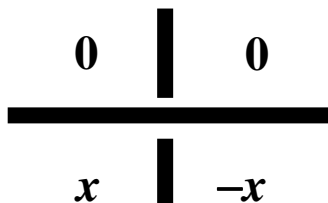
If we wanted to extend the range of knots we can handle without having to use negative numbers we could always start with 20, 20, 19 and 21 at the first crossing. The choice of 10, 10, 9 and 11 was simply a compromise between using negative numbers and having numbers too large.

§8.2. A High School Level Account

With high school students there's no need to avoid negative numbers. In fact it's good practice in using them. So for a presentation at this level we could use the numbers 0, 0, 1 and -1 at the first crossing. Also, since high-school students know a little algebra, we introduce



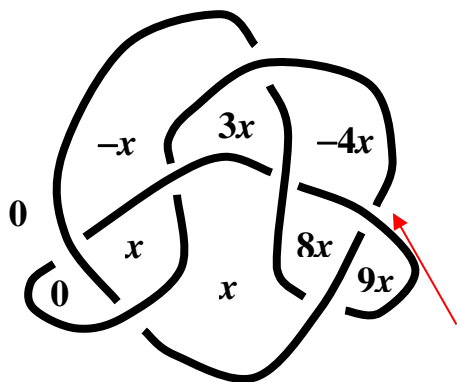
the variable x and use $0, 0, x$ and $-x$ at the first crossing.



This means that each space will contain a multiple of x . At the last crossing, when all four spaces are already filled in, the balancing equation will give a multiple of x equal to 0 . Rather than calling this a discrepancy, or a number of magic buttons needed to balance the crossing, at this level we write down an equation like $7x = 0$.

Here we'd need to explain why we can't conclude that $x = 0$, as we would in ordinary algebra. We'd need to say something about modulo arithmetic. But, at the end of the day, the students would simply write down the coefficient of x as the Alexander number. (If they wrote the equation as $-7x = 0$ we'd encourage them to simplify this to $7x = 0$ and write down 7 as the Alexander number

rather than -7 . The Alexander number is always positive.)



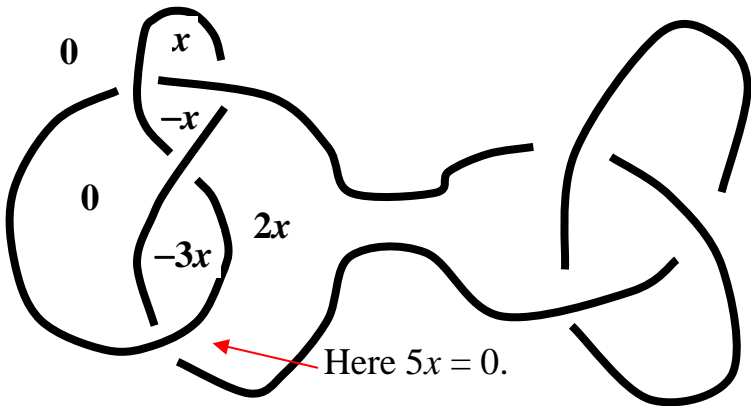
Example 5: Re-doing example 3 at this level:

last crossing

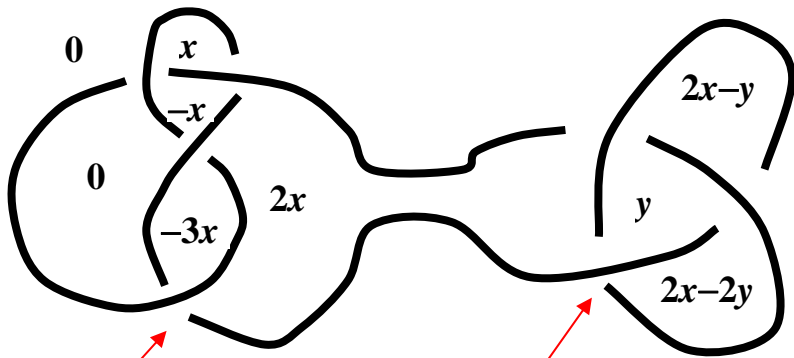
At the last crossing, $17x = -4x$ so $21x = 0$.
 The Alexander number is 21.

The introduction of x 's may seem quite unnecessary. However this is in preparation for more complicated knots. So far we've always been able to continue, until we reach the final crossing. We could always find a crossing that's ready to be balanced – where we have three out of the four regions labelled. What if that isn't the case? Consider the following example.

Example 6:



But we haven't finished labelling the remaining regions. All we do when we reach such an impasse is to label one of the remaining regions with a new variable, such as y . Then we continue as before. If we reach another impasse we'd have to introduce z and so on.



Here $5x = 0$.

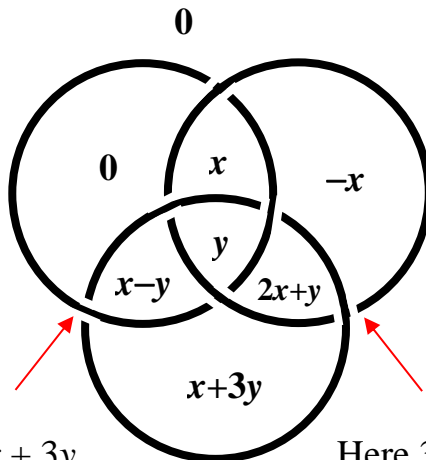
Here $2x + y = 2x - 2y$ so $3y = 0$.

We have two variables but we have two equations. Here we're lucky enough to get equations in which the variables occur separately. In this case we tell the students to simply multiply the coefficients to get the Alexander number. The Alexander number of this knot is therefore 15. (If they ask "why?" we have to fob them off. After all, we're not attempting to prove anything at this level. You'll get all the proof you want when we reach the final level.)

You'll notice that this knot is the sum of the figure 8 knot and a trefoil knot. These have Alexander numbers 5 and 3 respectively, and their sum has Alexander number 15. This is no accident. We can show that the Alexander number of the sum of two knots is the product of their Alexander numbers.

Our procedure, at this level, is to start with a balanced crossing $0, 0, x$ and $-x$. Then we continue balancing crossings as long as we have crossings with three out of the four regions labelled. If we can't proceed, we introduce the label y etc. At the end, if we've introduced n variables, we'll get n equations. If these variables occur separately, as in the above example, we simply multiply all these coefficients.

Example 7: Here we're jumping ahead a bit. This example is that of the Borromean Rings. It is not a knot because it isn't homeomorphic to a circle. Instead it has three components, all linked together. Such an object is called a **link**. We won't be studying links until the last chapter. However the Alexander Number is valid for links with any number of components.



Here $x - y = x + 3y$

Here $3x + 4y = -x$

Our two equations simplify down to:

$$\left. \begin{array}{l} 4y = 0 \\ 4x + 4y = 0 \end{array} \right\}$$

Here the variables aren't separate. Both occur in the second equation. However substituting from the first equation we can replace the second equation by $4x = 0$. Now we have our variables occurring separately and so we multiply the two coefficients to get 16 as the Alexander number of the Borromean Rings.

So we're permitted to use some algebraic manipulation in order to get equations that each contain only one variable. But we must be careful with what algebraic operations we use. We can substitute from one equation into another, or what amounts to the same thing we can add or subtract any multiple of one equation to or from another. What we're not allowed to do is to multiply or divide – otherwise we'd end up with $x = y = 0$ every time!

But what do we do if we can't manipulate our equations into single variable equations? In that case we need to use determinants – and that, of course, is out of the question at a high school level.

§ 8.3. A First Year University Level Account



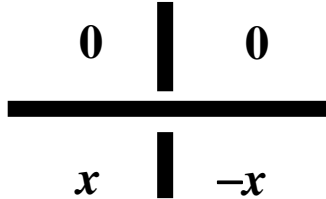
At this level we proceed in exactly the same way as we did at the high-school level. If we've had to introduce n variables we'll end up with n linear equations in these n variables. Of course we don't attempt to solve this system in the usual way, otherwise we'd end up with the unhelpful result: $x = y = \dots = 0$. Remember we're not dealing with real number algebra here.

What sort of algebra is it then? Well I'm being a bit vague on this at the moment. Just remember that, while you can add or subtract any equation to or from another you're not permitted to multiply or divide.

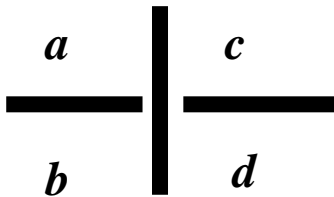
Instead you simply write the integer coefficients of the linear equations into a matrix, in the usual way, and take the determinant of this matrix. This will give an integer. It may be a negative integer, in which case you ignore the sign. The absolute value of this determinant will be the Alexander number of the knot.

The steps for finding the Alexander number of a knot are:

(1) Start with a balanced crossing $0, 0, x$ and $-x$. Make sure that the two 0 's are on the same side of the solid line, otherwise the crossing won't balance.



(2) Continue balancing crossings by filling in the fourth region where you have three already at the crossing. To balance a crossing you fill in the fourth region with whatever is needed to make the totals on each side of the unbroken line equal.



Balancing equation:

$$a + b = c + d$$

(3) Whenever you have unlabelled regions remaining, but none that can be balanced, there'll be at least one crossing where you have two out of the four regions labelled already. Introduce a new variable for one of the other regions there and balance to label the remaining one.

(4) If you introduced n variables then, once you've labelled every region, you'll have n crossings which don't balance. Write down the equations that occur at these

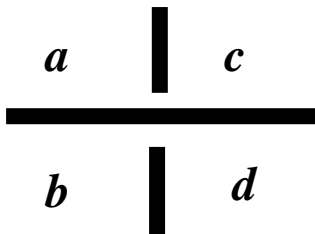
crossings, keeping the terms in the same order (eg $x, y, z \dots$).

(5) Write down the matrix of coefficients of these linear equations.

(6) Evaluate the determinant of this square matrix and take the absolute value. This integer is the Alexander number of the knot.

A couple of pieces of advice:

(1) Pay particular attention to the unbroken line. Mistakes occur because students ‘balance’ a crossing the wrong way.



**Incorrect
balancing**

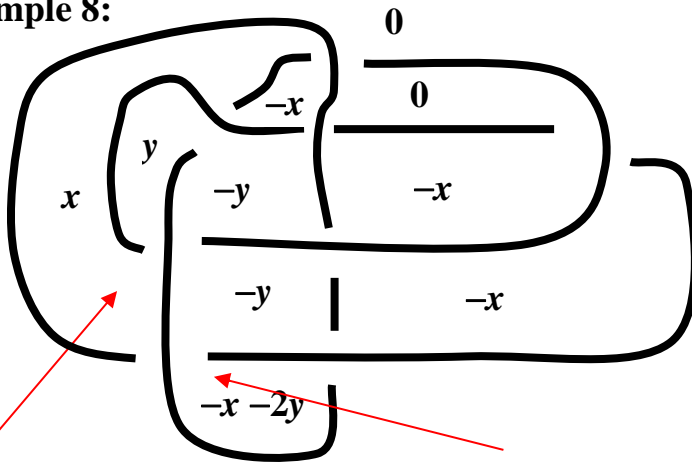
$$a + b = c + d$$

(2) Mark the crossings that you’ve balanced (perhaps by drawing a small circle around the crossing).

(3) Use fairly large diagrams to accommodate the algebraic expressions that you have to write in the spaces. If there’s insufficient space you can write the expression on the outside and indicate the region it refers to by an arrow.

(4) Don't forget that once a region has been given a label this operates at all crossings that surround that region. One common error is to overlook the fact that a region has already been labelled. This applies particularly to the outside. Once you label it '0' this applies to all crossings that involve the outside of the knot.

Example 8:



Here $x + y = -2y$
so $x + 3y = 0$.

Here $x = -x - 3y$
so $2x + 3y = 0$.

We have the system of equations:

$$\left. \begin{array}{l} x + 3y = 0 \\ 2x + 3y = 0 \end{array} \right\}$$

The matrix of coefficients is $\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ and its determinant is:

$$\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3.$$

The Alexander number of this knot is therefore 3.

Notice that this knot has the same Alexander number as the trefoil knots. But it isn't equivalent to either of them. You might like to convince yourself of this fact by making a rope model and trying to simplify it to a knot with just three crossings. It's impossible, but of course no amount of trying will prove that it's impossible. For this we need a stronger invariant, the Alexander polynomial. But that's for a later chapter.

You're now in a position to find the Alexander Number of any knot. Don't lose sight of what it means. It's a number that's associated with that knot, and knots that are equivalent to one another have the same Alexander number. Different Alexander numbers will prove that the knots are not equivalent. But having the same Alexander number means nothing, as we saw above.

I say "prove" but, of course, we've proved nothing in this chapter. It's all a bit of hocus-pocus at this stage, with magic buttons and mysterious balancing equations. We've yet to provide an account of all this at the Third Year University level, where you won't be satisfied until you've seen a proof that the number obtained in this way is an invariant. This we'll do shortly.

But it turns out that a better object to be using, as our invariant is not a number but rather an abelian group. Never mind if you don't know what that is. The next chapter is devoted to bringing you up to speed on this.

The Alexander Group is an abelian group, that is, a mathematical structure. For knots it's finite and the Alexander Number is simply the order, or size, of this mathematical structure. So we leave knots aside for a while and launch into the mysteries of finitely generated abelian groups.

EXERCISES AND SOLUTIONS FOR CHAPTER 8

In the appendix there is a table of all knots with Crossing Number up to 7. They are numbered K3, K4, K5.1, K5.2, K6.1, K6.2, K6.3 and K7.1 to K7.7. The first number is the Crossing Number and the second, where present, is the arbitrary number in the list. Below the name of the knot is the Alexander Number. Your task is to verify the given Alexander Numbers.